Non-minimally coupled scalar fields and isolated horizons

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Abstract

The isolated horizon framework is extended to include non-minimally coupled scalar fields. As expected from the analysis based on Killing horizons, entropy is no longer given just by (a quarter of) the horizon area but also depends on the scalar field. In a subsequent paper these results will serve as a point of departure for a statistical mechanical derivation of entropy using quantum geometry.

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I. INTRODUCTION

The notion of a weakly isolated horizon [1, 2, 3, 4, 5] extracts from Killing horizons the minimal properties required to establish the zeroth and the first laws of black hole mechanics. Thus, the notion captures the idea that the horizon itself is in equilibrium, allowing for dynamical processes and radiation in the exterior region. The resulting isolated horizon framework has had several applications: i) For fundamental physics, in addition to extending black hole mechanics from stationary situations [6, 7], it has led to a phenomenological model of hairy black holes [8, 9] which accounts for many of their key qualitative features; ii) In computational relativity, it has provided tools to extract physics from numerical simulations of gravitational collapse and black hole mergers at late times [1, 10, 11]; and, iii) For quantum gravity, it provided a point of departure for a statistical mechanical calculation of entropy, based on quantum geometry [12].

These applications feature gravity *minimally* coupled to matter such as scalar fields, Maxwell fields, dilatons, Yang-Mills fields and Higgs fields [3, 4, 8, 13]. A common element in all these diverse situations is that the first law always takes the form,

$$\delta E = \frac{1}{8\pi G} \, \kappa \delta a_\Delta + \text{work}$$

suggesting that a multiple of the surface gravity κ should be interpreted as the temperature and a multiple of the horizon area a_Δ as entropy. This is striking because, irrespective of the choice of matter fields *and their couplings to each other*, the entropy depends *only* on a single geometrical quantity, a_Δ, and is independent of the values of matter fields or their charges at the horizon. On the other hand, using Killing horizons, Jacobson, Kang and Myers [14] and Iyer and Wald [15] have analyzed general classes of theories, showing that the situation would be qualitatively different if the matter is non-minimally coupled to gravity. Now, *the expression of entropy depends also on matter fields on the horizon.*

Specifically, consider a scalar field ϕ coupled to gravity through the action

$$\mathcal{S}[g_{ab}, \phi] = \int \mathrm{d}^4 x \sqrt{-g} \left[\frac{1}{16\pi G} f(\phi) R - \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi - V(\phi) \right] , \tag{1.1}$$

where R is the scalar curvature of the metric g_{ab} and V is a potential for the scalar field. Then, the analysis of [14, 15] predicts that the entropy is given by

$$S = \frac{1}{4G\hbar} \left(\oint f(\phi) \, \mathrm{d}^2 V \right) . \tag{1.2}$$

where the integral is taken on any 2-sphere cross-section of the horizon. If the scalar field takes a constant value ϕ_0 on the horizon, the proportionality to area *is* restored, $S = [f(\phi_0)/4\ell_{\mathrm{P}}^2] \, a_\Delta$, but even in this case the constant now depends on ϕ_0. Thus, non-minimal coupling introduces a *qualitative* difference.

It is natural to ask if the isolated horizon framework can incorporate such situations. Apart from extending that framework, the incorporation would also serve three more specific purposes. First, we will have a richer class of examples. In particular the analysis based on Killing horizons requires a globally defined Killing field which admits a bifurcate horizon. While such solutions admitting scalar hair are known to exist if the cosmological constant Λ is non-zero [16], analytic work [17] and numerical evidence [18] suggests that

such solutions do not exist if Λ vanishes. The analysis [19] of the initial value problem based on isolated horizons, on the other hand, can be used to show that solutions admitting weakly isolated horizons would exist at least locally, whence the isolated horizon analysis would not trivialize in the $\Lambda = 0$ case. The second point is more technical. In [14, 15], a large class of theories is considered but under the assumption that the action depends only on the metric, the curvature and matter fields; first order actions which depend also on the gravitational connection are not incorporated. The isolated horizon analysis [3, 4], on the other hand, is based on first order actions. Hence, incorporation of non-minimal couplings in this framework would add to the robustness of the final results of [14, 15]. Finally, the analysis based on Killing horizons does not provide an action principle or a Hamiltonian framework which can be used for a non-perturbative quantization. The isolated horizon framework does, thereby paving the way for a fully statistical mechanical treatment based on quantum gravity.

The purpose of this paper is to extend the isolated horizon framework to incorporate non-minimally coupled scalar fields of Eq (1.1). We will find that the entropy is indeed given by (1.2); the main result of [14, 15] is robust. In a subsequent paper we will show that this analysis provides the point of departure for quantum theory and the fully quantum mechanical calculation assigns the same entropy to the isolated horizon.

Since the couplings of matter fields among themselves do not play a direct role in our analysis, in the rest of the paper the term 'non-minimal coupling' will refer to the couplings of matter fields to gravity.

II.   NON-MINIMALLY COUPLED FIELDS IN THE FIRST ORDER FORMALISM

We will first recall the second and first order actions of interest and then specify the first order action that will be used in the rest of the paper. For simplicity, in the first part we will omit surface terms and restore them only at the end.

Let us then start with the action:

$$\mathcal{S}[g_{ab}, \phi] = \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} \left[\frac{1}{16\pi G} f(\phi) R - \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi - V(\phi) \right] \qquad (2.1)$$

on a 4-dimensional manifold $\mathcal{M}$. In general relativity, one is often interested in the case $f(\phi) = 1 + 8\pi G \xi \phi^2$ with ξ a constant. Then ϕ satisfies a non-minimally coupled equation $\Box \phi + \xi R \phi - \partial V(\phi)/\partial \phi = 0$. However, our analysis is not tied to this case.

As indicated in section I, in the isolated horizon framework it is simpler to work in the first order formalism. The first order action for non-minimally coupled scalar fields was discussed in [20, 21]. Let us begin by recalling the main difference from the simpler, minimally coupled theory. In the case of minimal couplings, the first order action is given just by replacing the scalar curvature term in the second order action by an appropriate contraction of the tetrads with the curvature of the connection [22]. In the present case, there is an additional term because derivatives of the scalar field appear when one integrates by parts to obtain the equations of motion of the connection. More precisely, if we work with tetrads e_I^a and Lorentz connections $A_{aI}{}^J$, the first order action is given by:

$$\mathcal{S}[e, A, \phi] = \int_{\mathcal{M}} \mathrm{d}^4 x \, e \left[\left(\frac{1}{16\pi G} f(\phi) \, e_I^a e_J^b F(A)_{ab}^{IJ} \right) - \frac{1}{2} K(\phi) \partial_a \phi \partial_b \phi \, e_I^a e_J^b \eta^{IJ} - V(\phi) \right],$$
$$(2.2)$$

where

$$K(y) = [1 + (3/16\pi G)(f'(y))^2/f(y)].$$ (2.3)

To show that this action is equivalent to the second order action (2.1), it suffices to solve the equation of motion for the connection A, substitute the solution in (2.2) and show that the result reduces to (2.1).

Variation with respect to A yields the equation of motion for the connection:

$$D_a \left(f(\phi) \, e \, e^a_{[I} e^b_{J]} \right) = 0.$$ (2.4)

Assuming that the *rescaled* tetrad $\hat{e}^a_I = (p\,e^a_I)$, with $p = 1/\sqrt{f(\phi)}$, is well-defined and non-degenerate, it follows that A is the unique Lorentz connection compatible with $\hat{e}^a_I$. Substituting this solution in the expression of the curvature, we obtain:

$$e^a_I e^b_J F^{IJ}_{ab}(A) \;=\; \left(\frac{1}{p^2} \right) \hat{e}^a_I \hat{e}^b_J F^{IJ}_{ab}(A) \;=\; f(\phi)\hat{R}$$ (2.5)

where $\hat{R}$ is the scalar curvature of the metric $\hat{g}_{ab} = \hat{e}^I_a \hat{e}^J_b \eta_{IJ}$. Hence, in terms of the metric $g_{ab} = e^I_a e^J_b \eta_{IJ}$, the action (2.2) reduces to the second order form

$$\mathcal{S}[g,\phi] \;=\; \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g} \left[\left(\frac{1}{16\pi G} \right) f(\phi)\hat{R} - \frac{1}{2}K(\phi)g^{ab}\partial_a\phi\partial_b\phi - V(\phi) \right]$$ (2.6)

Finally, using the standard relation between the scalar curvatures of g_{ab} and $\hat{g}_{ab}$, we recover, up to a surface term, the second order action (1.1) we began with.[1] Thus, when $\hat{e}^a_I$ is smooth and non-degenerate, the first order action (2.2) is equivalent to the more familiar one. However, we wish to emphasize that, for our purposes, it is the first order action that is fundamental and this action as well as the Hamiltonian framework developed in this paper continue to be well-defined even when $f(\phi)$ vanishes or e^a_I becomes degenerate.

To make contact with literature on isolated horizons, it will be convenient to rewrite the first order action (2.2) in terms of forms. Let us define the two form

$$\Sigma^{IJ} := \frac{1}{2}\epsilon^{IJ}{}_{KL}\, e^K \wedge e^L$$

where e^I_a are the co-tetrads so that $e^a_I e^J_b = \delta^a_b \delta^J_I$. The action now takes the form,

$$\mathcal{S}[e,A,\phi] \;=\; \int_{\mathcal{M}} \left(\frac{1}{16\pi G} f(\phi)\,\Sigma^{IJ} \wedge F_{IJ} + \frac{1}{2}K(\phi)\,{}^\star\mathrm{d}\phi \wedge \mathrm{d}\phi - V(\phi)\,{}^4\epsilon \right)$$
$$- \frac{1}{16\pi G} \int_{\partial\mathcal{M}} A^{IJ} \wedge \Sigma_{IJ},$$ (2.7)

where * denotes the Hodge-dual, $^4\epsilon$ is the volume 4-form on $\mathcal{M}$ defined by the tetrad e^a_I and where we have explicitly included the surface term that is needed to make the action differentiable. In the remainder of this paper we shall use this form of the action.

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[1] Since in the isolated horizon framework the horizon is treated as a physical boundary, the surface term has to be examined carefully. It is proportional to $\oint \mathrm{d}S^a \partial_a f$. We will see in the next section that on an isolated horizon ϕ is 'time independent'. It then follows that the horizon contribution to the surface term vanishes whence the two action principles are in fact equivalent.

We conclude with two remarks on the second order action.

1. Consider again the second order action (2.1). In the sector of the theory in which $f(\phi)$ is nowhere zero, we can pass to a conformally related metric $\bar{g}_{ab} = f(\phi)g_{ab}$ and to a new field $\varphi = F(\phi)$ where F is defined by

$$F(x) = \int^x \left[\frac{1}{f(y)} + \frac{3}{16\pi G} \left(\frac{f'(y)}{f(y)} \right)^2 \right]^{1/2} \mathrm{d}y \,. \tag{2.8}$$

Then the action (2.1) can be rewritten as:

$$\mathcal{S}[\bar{g}_{ab}, \varphi] = \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-\bar{g}} \left[\frac{1}{16\pi G} \bar{R} - \frac{1}{2} \bar{g}^{ab} \partial_a \varphi \partial_b \varphi - v(\varphi) \right] \tag{2.9}$$

where the potential for the scalar field φ is given by $v(\varphi) = (1/f^2(\phi(\varphi)))V(\phi(\varphi))$. Thus, on this sector, the theory is equivalent to a minimally coupled scalar field φ on $(M, \bar{g})$. In the commonly used terminology, (2.1) expresses the theory in the Jordan conformal frame while (2.9) expresses it in the Einstein frame.

2. How stringent is the restriction that $f(\phi)$ be everywhere positive? The equations of motion following from (2.1) are:

$$0 = g^{ab} \nabla_a \nabla_b \phi + \frac{1}{16\pi G} R \frac{\partial f(\phi)}{\partial \phi} + \frac{\partial V(\phi)}{\partial \phi}$$

$$\frac{f(\phi)}{8\pi G} \left(R_{ab} - \frac{R}{2} g_{ab} \right) = \nabla_a \phi \nabla_b \phi - \left(\frac{1}{2} \nabla^c \phi \nabla_c \phi - \nabla^c \nabla_c f(\phi) + V(\phi) \right) g_{ab}$$
$$+ \nabla_a \nabla_b f(\phi) \tag{2.10}$$

Therefore, if $f(\phi)$ were to vanish in an open set, the Einstein tensor is undetermined there. Consequently, it is likely that the Cauchy problem would not be well-posed. Indeed, if the potential admits a minimum at $\phi = k$, a constant, then on an open set on which $f(\phi)$ vanishes, $\phi = k$, and g_{ab} *any* metric with $R = 0$ would satisfy the field equations. By contrast, if $f(\phi)$ is nowhere zero, as we just saw, the field equations are equivalent to those of minimally coupled scalar field and the Cauchy problem is then well-posed. Therefore, from the standard viewpoint one adopts in general relativity, the requirement that $f(\phi)$ does not vanish on an open set is physically quite reasonable.

III. BOUNDARY CONDITIONS AND ACTION PRINCIPLE

Let us first adapt the basic definitions to accommodate non-minimal couplings.

A *non-expanding horizon* Δ is a null, 3-dimensional sub-manifold of $(\mathcal{M}, g_{ab})$, topologically $S^2 \times R$ such that:

i) The expansion $\Theta_{(\ell)}$ of every null normal ℓ to Δ vanishes;

ii) The scalar field ϕ satisfies $\mathcal{L}_\ell \phi \,\widehat{=}\, 0$; and

iii) Equations of motion hold on Δ.

Here and in what follows $\widehat{=}$ denotes equality restricted to points of Δ. The previous papers on isolated horizons assumed, in place of ii), that the matter stress-energy satisfies a very weak energy condition which, through the Raychaudhuri equation, implied that matter fields

are Lie dragged by ℓ^a. Non-minimally coupled scalar fields violate even that energy condition. Therefore, we directly assume ii) which, it turns out, suffices for our purposes. (This is the only change in the basic definitions needed to accommodate non-minimal couplings.) This condition captures the intuitive idea that, since the horizon Δ is in equilibrium, the scalar field should be time-independent on Δ. The definition also implies that the intrinsic (degenerate) metric q_{ab} on Δ is time independent; $\mathcal{L}_\ell q_{ab} \,\widehat{=}\, 0$.

As in the minimally coupled case, the space-time covariant derivative ∇ induces a natural derivative operator $\mathcal{D}$ on Δ such that $\mathcal{D}_a q_{bc} \,\widehat{=}\, 0$ and $\mathcal{D}_a \ell^b \,\widehat{=}\, \omega_a \ell^b$ for some 1-form ω_a on Δ. While $\mathcal{D}$ is canonical, the 1-form ω_a depends on the choice of the (future-directed) null normal; under rescaling $\ell^a \to \tilde{\ell}^a = f\ell^a$ we have $\tilde{\omega}_a = \omega_a + \mathcal{D}_a \ln f$.

A *weakly isolated horizon* $(\Delta, [\ell])$ is a pair consisting of a non-expanding horizon Δ, equipped with an equivalence class of null normals ℓ^a satisfying
$$\mathcal{L}_\ell \omega_a \,\widehat{=}\, 0,$$
where $\ell \approx \ell'$ if and only if $(\ell')^a = c\ell^a$ for a positive constant c. As in [3, 4], to establish the zeroth and the first law of black hole mechanics, we will not need the stronger notion of isolated horizons.

As in [3, 4], an immediate consequence of these boundary conditions is the zeroth law. Since $\mathcal{L}_\ell \omega_a \,\widehat{=}\, 0$, by the Cartan identity, we have:

$$2\ell^a \mathcal{D}_{[a}\omega_{b]} + \mathcal{D}_b(\omega_a \ell^a) \,\widehat{=}\, 0 \tag{3.1}$$

For reasons discussed in detail in [2], $D_{[a}\omega_{b]}$ is proportional to ϵ_{ab}, the natural 2-volume element on Δ satisfying $\epsilon_{ab}\ell^b \,\widehat{=}\, 0$. Hence, we conclude

$$\mathcal{D}_a \kappa_\ell := \mathcal{D}_a(\omega_a \ell^a) \,\widehat{=}\, 0 \quad \text{i.e.} \quad \kappa_\ell \,\widehat{=}\, \text{const.} \tag{3.2}$$

To formulate the action principle, we fix a manifold $\mathcal{M}$ bounded by two (would be) space-like, partial Cauchy surfaces $M^\pm$, an internal boundary Δ which is topologically $S^2 \times R$ (the would be weakly isolated horizon). (See Figure 1.) We will choose a fixed equivalence class of vector fields $[\ell_0^a]$ along the 'R direction' of Δ. It is convenient also to fix a flat tetrad and a connection at infinity and an *internal* Minkowski metric η_{IJ} and an *internal* null tetrad $\ell^I, n^I, m^I, \bar{m}^I$ on Δ. A *history* will consist of an orthonormal tetrad e_I^a, a Lorentz connection A_a^{IJ} and a scalar field ϕ on $\mathcal{M}$ such that:
i) $M^\pm$ are space-like, partial Cauchy surfaces on $(M, g_{ab} := \eta_{IJ}\, e_a^I e_b^J)$;
ii) $e_I^a \ell^I \in [l_0^a]$ on Δ;
iii) $(\Delta, [\ell])$ is a weakly isolated horizon; and,
iv) the fields satisfy the standard asymptotic flatness conditions at spatial infinity. (For further details, see [3] or [4].)

In the variational principle, we fix the fields[2] e_I^a, A_a^{IJ} and ϕ on $M^\pm$. Since the boundary conditions require $\mathcal{L}_\ell \phi \,\widehat{=}\, 0$ and $\mathcal{L}_\ell \omega \,\widehat{=}\, 0$, in each history that features in the variation, $\delta\phi \,\widehat{=}\, 0$ and $\delta\omega_a \,\widehat{=}\, 0$. We will now use this information to show that the first order action (2.7) leads to a well-defined variational principle.

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[2] Since we are working with a first order framework, we will extend the action of the derivative operator ∇ to fields also with internal indices. Then $\nabla_a V_I = \partial_a V_I + A_{aI}{}^J V_J$, where the flat derivative operator ∂ on internal indices will be assumed to be compatible with (i.e. annihilate) the internal tetrad on Δ. Thus the gauge freedom *at* Δ is restricted.
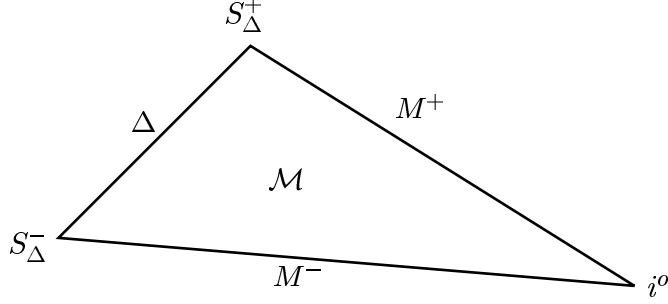
FIG. 1: The region of space-time $\mathcal{M}$ under consideration has an internal boundary Δ and is bounded by two partial Cauchy surfaces $M^{\pm}$ which intersect Δ in the 2-spheres $S_{\Delta}^{\pm}$ and extend to spatial infinity i^o.

If we denote the variables e_a^I, A^{IJ}, ϕ collectively as Ψ, we have:

$$\delta\mathcal{S}[\Psi] = \int_{\mathcal{M}} E[\Psi]\,\delta\Psi + \int_{\partial\mathcal{M}} J[\Psi,\delta\Psi] \tag{3.3}$$

where $E[\Psi] = 0$ is the equation of motion for Ψ and the current 3-form J is given by:

$$J[\Psi,\delta\Psi] = \frac{1}{16\pi G}\left(f(\phi)\,\Sigma^{IJ}\wedge\delta A_{IJ}\right) + K(\phi)\,{}^*\mathrm{d}\phi\wedge\delta\phi \tag{3.4}$$

Thus, the action principle is well-defined if and only if the integral of J over the boundary of $\mathcal{M}$ vanishes. Now, since all fields are kept fixed on $M^{\pm}$, J itself vanishes there. Similarly, the boundary conditions at spatial infinity ensure, as usual, that the boundary term at infinity also vanishes. Therefore, we only need to check the boundary integral on the horizon Δ. Since $\delta\phi\,\hat{=}\,0$, the second term in J vanishes. To evaluate the first term, let us first calculate δA_b^{IJ} on Δ. We first note that A_a^{IJ} can be expressed in terms of tetrads and ω_a: Since

$$\nabla_b\ell^a = \nabla_b(e_I^a\ell^I) = (\nabla_b e_I^a)\ell^I - e_I^a A_b^{IJ}\ell_J\,,$$

using the equation of motion $\nabla_a \hat{e}_I^b = \nabla_a(p\,e_I^b)\,\hat{=}\,0$ which holds on Δ (because the definition of a non-expanding horizon ensures that the field equations hold on Δ), we have

$$A_b^{IJ}\,\hat{=}\,2(\nabla_b\ln p)\ell^{[I}n^{J]} + 2\omega_a\ell^{[I}n^{J]} + C_b^{IJ} + \ell_b U^{IJ} \tag{3.5}$$

for some C_a^{IJ} and U^{IJ} satisfying $C_a^{IJ}\ell_J\,\hat{=}\,0$ and $U^{(IJ)}\,\hat{=}\,0$. Now, since the internal tetrad is fixed once and for all on Δ, independently of the choice of the history under consideration, variations of these internal vectors vanishes. Similarly, since ℓ_b is the unique direction field (in the cotangent space at any point of Δ) which is orthogonal to all tangent vectors to Δ in *any* history, $\delta\ell_b$ is necessarily proportional to ℓ_b. Therefore, the variation δA_b^{IJ} is given by:

$$\delta A_b^{IJ}\,\hat{=}\,2(\nabla_b\,\delta\ln p + \delta\,\omega_b)\,\ell^{[I}n^{J]} + \delta C_a^{IJ} + \delta U^{IJ}\ell_b \tag{3.6}$$

We can now substitute (3.6) in the surface term in δS. Using the fact that ℓ^a is a null normal to Δ and the algebraic properties of C_a^{IJ} and U^{IJ} on Δ, we obtain:

$$\int_{\Delta} J\,\hat{=}\,\frac{1}{8\pi G}\int_{\Delta} f(\phi)\,(\mathrm{d}(\delta\ln p) + \delta\omega)\wedge\epsilon = 0 \tag{3.7}$$

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where, in the last step, we have used $\delta\phi \mathrel{\hat{=}} 0$ and $\delta\omega \mathrel{\hat{=}} 0$. Thus the surface term in (3.3) vanishes because of the boundary conditions, whence the action (2.7) is functionally differentiable. From discussion of section II we know that the equations of motion $E[\Psi] = 0$ are the expected ones.

IV. PHASE SPACE AND THE FIRST LAW

As in the previous papers on isolated horizons, we will use a covariant phase space. Thus, our phase space Γ will consist of *solutions* (e_a^I, A_a^{IJ}, ϕ) to the field equations on $\mathcal{M}$ which are asymptotically flat and admit $(\Delta, [\ell^a])$ as a weakly isolated horizon. (The precise kinematical structure fixed on Δ is listed in Section III.) As usual, the symplectic current is constructed from the anti-symmetrized second variation of the action, where one can now use the equations of motion *but the variations are no longer restricted to vanish on the initial and final surfaces $M^\pm$*. To express the resulting symplectic structure in a convenient form, it is convenient to introduce a scalar potential ψ of surface gravity κ_ℓ on Δ via:

$$\ell^a \nabla_a \psi = \kappa_\ell, \quad \text{and} \quad \psi\,|_{S_\Delta^-} = 0 \tag{4.1}$$

where, as in fig 1, S_Δ^- is the 2-sphere on which Δ intersects M^-. (Note that ψ is independent of the choice of vector ℓ^a within the equivalent class $[\ell]$.) Then, the symplectic structure is given by:

$$
\begin{aligned}
\Omega(\delta_1, \delta_2) = \quad & \frac{1}{16\pi G} \int_M \mathrm{Tr}\left[\delta_1(f\Sigma) \wedge \delta_2 A - \delta_2(f\Sigma) \wedge \delta_1 A\right] \\
& + \int_M K(\phi)\left[(\delta_1\phi)\,\delta_2({}^\star\mathrm{d}\phi) - (\delta_2\phi)\,\delta_1({}^\star\mathrm{d}\phi)\right] \\
& + \oint_{S_\Delta} \left[\delta_1(f\epsilon)\,\delta_2\psi - \delta_1(f\epsilon)\,\delta_2\psi\right],
\end{aligned}
\tag{4.2}
$$

where M is any partial Cauchy surface which intersects Δ in S_Δ. Thus, as in the minimally coupled case [3, 4], the symplectic structure has a surface term which again comes from the 'gravitational part' of the action — the first term in (2.7)— but now depends also on the value of the scalar field on Δ.

To obtain the first law, as in [4] we need to make an additional assumption: We will restrict ourselves to the case when the weakly isolated horizons are of symmetry type II, i.e., *axi-symmetric*. Thus, we will fix a rotational vector field R^a on Δ (normalized so its closed orbits have affine length equal to 2π) and restrict ourselves to the part of the phase space where the fields (q_{ab}, ω_a, ϕ) are all Lie-dragged by R^a. Note that the symmetry restriction is imposed *only* at Δ; there is no assumption that the fields in the bulk are axi-symmetric.

Following [4], let us first define the horizon angular momentum. Consider any extension $\tilde{R}^a$ of R^a on Δ to the bulk space-time $\mathcal{M}$ of which tends to an asymptotic rotational Killing field at spatial infinity. The question is whether motions along $\tilde{R}^a$ induce canonical transformations on the phase space and, if so, what its generating function $J_{\tilde{R}}$ is. The simplest way to analyze this issue is to set $\delta_{\tilde{R}} = (\mathcal{L}_{\tilde{R}} e, \mathcal{L}_{\tilde{R}} A, \mathcal{L}_{\tilde{R}} \phi)$, and ask whether there exists a phase space function $J^{\tilde{R}}$ such that

$$\Omega(\delta, \delta_{\tilde{R}}) = \delta J^{\tilde{R}} \tag{4.3}$$

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for all tangent vectors δ to the phase space. The analysis is completely analogous to that of [4]. However, since one has to use both the field equations obeyed by (e, A, ϕ) and the linearized field equations satisfied by $(\delta e, \delta A, \delta \phi)$, the calculation is significantly more complicated because these equations are more involved now. However, the final result is rather simple:

$$\Omega(\delta, \delta_{\tilde{R}}) = \frac{1}{16\pi G} \left[\oint_{S_\infty} \delta \mathrm{Tr}(\tilde{R}^a A_a \, f\epsilon) - 2 \oint_{S_\Delta} \delta(R^a \omega_a \, f\epsilon) \right] \tag{4.4}$$

where ϵ is the volume 2-form on the 2-sphere under consideration. Hence, as is usual in generally covariant theories, $J_{\tilde{R}}$ consists only of surface terms. The term at infinity can be shown to be the total angular momentum associated with $\tilde{R}^a$. It is then natural to interpret the surface integral at Δ as the isolated horizon angular momentum:

$$J_\Delta^R = -\frac{1}{8\pi G} \oint_{S_\Delta} R^a \omega_a \, f\epsilon \,. \tag{4.5}$$

Thus, the only difference from the minimally coupled case [4] is that volume 2-form ϵ there is now replaced by $f\epsilon$.

We are now ready to obtain the first law. To introduce the notion of energy, we have to consider vector fields t^a in space-time $\mathcal{M}$ representing time translations. Since we are dealing with a generally covariant theory, what matters is only the boundary values of t^a. It is clear that t^a must approach an asymptotic time translation at infinity and reduce to a symmetry vector field representing time translations on the horizon:

$$t^a \,\hat{=}\, B_{(\ell,t)} \ell^a - \Omega_{(t)} R^a \tag{4.6}$$

where $B_{(\ell,t)}$ and $\Omega_{(t)}$ are constants on the horizon and, as the notation suggests, B depends not only on t^a but also on our choice of the null normal ℓ^a in $[\ell^a]$ (such that $B_{(\ell,t)} \ell^a$ is unchanged under the rescalings of ℓ in $[\ell]$.) At infinity, all metrics tend to a universal flat metric whence the asymptotic values of the symmetry vector fields are also universal. At the horizon, on the other hand, we are in a strong field region and the geometry is not universal. Therefore, there is ambiguity in what one means by the 'same' symmetry vector field in two different space-times. For example, in the Kerr family, for the 'standard' time-translation, $\Omega_{(t)} \,\hat{=}\, 0$ in the Schwarzschild space-time but non-zero if the space-time has angular momentum. Therefore, a priori we must allow for the possibility that the horizon values of t^a (i.e., the constants $B_{(\ell,t)}, \Omega_{(t)}$) may vary from one space-time to another. In the numerical relativity terminology these are 'live' vector fields. This subtlety has nothing to do with non-minimal couplings and also arose in all previous work on isolated horizons.

Again, the key question is whether there exists a phase space function E^t such that $\Omega(\delta, \delta_t) = \delta E^t$. Using the equations of motion and their linearized version, one can show that $\Omega(\delta, \delta_t) = \delta E^t$ consists of two surface terms. Let us focus on the one at the horizon. A long calculation yields:

$$\begin{aligned} \Omega(\delta, \delta_t) \,|_\Delta &= \frac{1}{8\pi G} \oint_{S_\Delta} \left[\kappa_{(t)} \delta(f\epsilon) - \Omega_{(t)} \delta(R^a \omega_a) \, f\epsilon \right] \\ &= \left[\frac{\kappa_{(t)}}{8\pi G} \delta \oint_{S_\Delta} f \, \epsilon \right] + \left[\Omega_{(t)} \delta J_\Delta^R \right] \end{aligned} \tag{4.7}$$

where $\kappa_{(t)}$ is the surface gravity (i.e. acceleration) of the vector field $B_{(\ell,t)}\ell^a$ on Δ. (The surface term at infinity is precisely $\delta E_{\text{ADM}}^{(t)}$.) Thus, while the diffeomorphisms generated by t^a give rise to a flow on the covariant phase space, in general this flow may *not* be Hamiltonian. It is so if and only if there exists a phase space function $E_\Delta^{(t)}$ such that

$$\delta E_\Delta^t = \left[\frac{\kappa_{(t)}}{8\pi G} \delta \oint_{S_\Delta} f\, \epsilon \right] + \left[\Omega_{(t)} \delta J_\Delta^R \right] , \tag{4.8}$$

i.e., if and only if the first law is satisfied. Thus, as in the case of minimal coupling [4], the first law arises as a necessary and sufficient condition for the flow generated by t^a to be Hamiltonian.

As in [4], one can show that there exist infinitely many vector fields t^a for which the right side of (4.8) is an exact variation and provide an explicit procedure to construct them. Each of these vector fields generates a Hamiltonian flow and gives rise to a first law. Thus, the overall structure is the same as in the case of minimal coupling. However, there is a key difference in the expression of the multiple of $\kappa_{(t)}/8\pi G$: while it is the variation in the horizon area a_Δ in the case of minimal coupling, now it is the variation of the integral of f on a horizon cross-section. Consequently, the entropy is now given by (1.2). As in [4], $E_\Delta^{(t)}$ and $\kappa_{(t)}$ depend on the choice of the time translation t^a, *while entropy does not*.

Finally, we note that if the horizon geometry fails to be axi-symmetric, there is no natural notion of angular momentum. However, we can still repeat the argument by seeking 'time-translation' vector fields t^a with $t^a \widehat{=} B_{(\ell,t)}\ell^a$, the evolution along which is Hamiltonian. Such vector fields exist. Thus, there is still a first law and the entropy still given by (1.2) and is again independent of the choice of t^a.

We will conclude with some remarks.

i) It may first appear that the surface integrals in the expressions (4.5) of the horizon angular momentum and (4.8) of the first law arise directly from the surface term in the symplectic structure (4.2). This is not the case. In fact in the detailed calculation the contributions of the surface term in (4.2) to $\Omega(\delta, \delta_{\tilde{R}})$ and $\Omega(\delta, \delta_t)$ vanish identically! The key surface terms in (4.5) and (4.8) come from integrations by part of the bulk term in the symplectic structure, required to relate the integrands to the field equations satisfied by (e, A, ϕ) and $(\delta e, \delta A, \delta \phi)$. The overall situation is the same as in [3] for minimal coupling but these calculations are now considerably more complicated.

ii) Conceptually, it may appear strange that the flow generated by t^a is not always Hamiltonian. However, this is in fact the 'rule' rather than the 'exception'. Consider the asymptotically flat situation without boundaries and suppose we allow 'live' vector fields. Now, a live asymptotic time-translation vector field t^a may point in the same direction at infinity but may have a norm which varies from space-time to space-time. (Indeed, it may even point in different directions in different space-times.) The flow generated by such vector fields in the phase space fails to be Hamiltonian in general; it is Hamiltonian if and only of the asymptotic value of the vector field is the same in all space-times; i.e., t^a defines the *same* time translation of the fixed flat metric at infinity in all space-times in the phase space. As mentioned above, in the case of the isolated horizon, it is not a priori clear what the 'same' time translation on Δ means. The pleasant surprise is that it is the first law that settles this issue.

iii) In the Einstein-Maxwell theory, one can exploit the black hole uniqueness theorem to select a *canonical* evolution field t^a on Δ of each space-time in the phase space [4]. There is then a canonical notion of horizon energy —which can be taken to be the horizon mass—

and a canonical first law. In the non-minimally coupled theory now under consideration, it is no longer true that there is precisely a 2-parameter family of globally stationary solutions. Hence, one can not select a canonical vector field or a canonical first law. Consequently, although the expression of entropy is unambiguous, that of horizon mass is not; we only have a t-dependent notion of the horizon energy E_Δ^t. Nonetheless, as with hairy black holes [9], one can extract physically useful information from these expressions.

iv) While we focused in this paper on 3+1 general relativity without cosmological constant, it is straightforward to incorporate 2+1 dimensional isolated horizons [24] and the presence of cosmological constant [2]. In particular, our results apply to the 2+1 dimensional black hole studied in [25], where, using Euclidean methods, entropy was found to be $\frac{2}{3}\left(a_\Delta/4G\hbar\right)$. The '2/3' factor, which was left as a puzzle, is naturally accounted for by the fact that entropy depends also on the scalar field via (1.2).

V. DISCUSSION

We have seen that, in presence of a weakly isolated horizon internal boundary, there is a well-defined action principle and Hamiltonian framework also for non-minimally coupled scalar fields. While the overall structure is the same as in the minimally coupled case, the first law is now modified, suggesting that now the entropy is given by (1.2). Thus we have extended the main result of Jacobson, Kang and Myers [14] and Iyer and Wald [15] from stationary space-times to those admitting only isolated horizons and shown that the result holds also in first order frameworks. This is however only a small extension because whereas [14] and [15] consider a very large class of theories, here we considered only non-minimally coupled scalar fields.

From the discussion of section II, it follows that we can make a conformal transformation of the tetrad e and a field redefinition of ϕ to cast action (2.7) in to that of the minimally coupled Einstein-scalar field theory with tetrad $\bar{e}_a^I = \sqrt{f(\phi)}e_a^I$ and scalar field φ related algebraically to ϕ (via (2.8).) It is then natural to ask if our results would have been the same if we had worked from the beginning in this minimally coupled 'Einstein frame' rather than the original non-minimally coupled 'Jordan frame'. One's first reaction may be that it is obvious that the notions of angular momentum, energy and entropy of a field configuration should not depend on whether we regard it as a solution to the first theory or the second, whence the results must be the same. However, this is not a priori clear. For example, it is not self-evident that $(\Delta, [\ell])$ is even a weakly isolated horizon for $(\mathcal{M}, \bar{g}_{ab})$. A more subtle point is that there could be tension because even when a state is shared by two theories, its properties *can* depend on the theory we choose to analyze it in. A striking example is provided by the magnetically charged Reissner-Nordstrom space-times: While they are stable in the Einstein-Maxwell theory, they are unstable when regarded as solutions to the Einstein-Yang-Mills equations [23]. In the isolated horizon framework, this difference can be directly traced to the fact that the *horizon* mass is 'theory dependent' [9]. More precisely, for a fixed magnetically charged Reissner-Nordstrom space-time, the horizon mass in the Einstein-Maxwell theory is the same as the ADM mass while that in the Einstein-Yang-Mills theory is lower. The difference can be radiated away, paving the way for an instability in the Einstein-Yang-Mills theory [9]. The isolated horizon framework even provides a qualitative relation between the horizon area and the frequency of unstable modes in this theory.

It is therefore worthwhile to compare the results obtained here in the Jordan frame with

those one would obtain in the Einstein frame. The main results can be summarized as follows:

- $(\Delta, [\ell])$ is a non-expanding horizon in $(\mathcal{M}, g_{ab})$ if and only if it is one in $(\mathcal{M}, \bar{g}_{ab} = f(\phi)g_{ab})$. This follows because $\mathcal{L}_\ell \phi \,\widehat{=}\, 0$.

- The intrinsic horizon metrics are related by $\bar{q}_{ab} = f q_{ab}$, and the volume 3-forms by $\bar{\epsilon} = f \,\epsilon$. For any choice of the null normal ℓ^a, $\bar{\omega}_a = \omega_a + \frac{1}{2} D_a \ln f$. Since the surface gravity is given by $\kappa_\ell = \omega_a \ell^a$, and $\mathcal{L}_\ell \phi \,\widehat{=}\, 0$ implies $\mathcal{L}_\ell f \,\widehat{=}\, 0$, the two surface gravities are equal: $\bar{\kappa}_\ell = \kappa_\ell$.

- In type II horizons, $R^a \bar{\omega}_a \,\widehat{=}\, R^a \omega_a$ because $\mathcal{L}_R \phi \,\widehat{=}\, 0$. As a result, angular momenta J_Δ^R computed in the Einstein and Jordan frames agree.

- Diffeomorphisms generated by a vector field t^a on $\mathcal{M}$ generate a Hamiltonian flow on the phase space in the Einstein frame if and only if there exists a phase space function $\bar{E}_\Delta^t$ such that

$$
\delta \bar{E}_\Delta^t = \left[\frac{\kappa_{(t)}}{8\pi G} \, \delta \bar{a}_\Delta \right] + \left[\Omega_{(t)} \delta J_\Delta^R \right] , \tag{5.1}
$$

  where $\bar{a}_\Delta$ is the horizon area in the Einstein frame. Since the two volume 2-forms are related by $\bar{\epsilon} = f\epsilon$, it follows that the right sides of (4.8) and (5.1) are identical, whence the values of the entropy and horizon energy calculated in the two frames are the same.

Thus, the main results are the same in the two frames; the situation is different from that with the Einstein-Maxwell and Einstein-Yang-Mills theories discussed above. This difference can be traced back to the fact that while the phase spaces of those two theories are quite different, *in the case when f is everywhere positive*, the phase spaces of the minimally and non-minimally coupled theories are naturally isomorphic.

Note however that the action (2.7), the boundary conditions at Δ, the symplectic structure (4.2), and the vector field δ_t on the phase space are all well-defined *also in the case when f vanishes on an open set* of compact closure away from Δ, so long as $K(\phi)$ remains everywhere smooth.[3] Therefore, in the first order framework used here, the derivation of the first law in the Jordan frame goes through even when one can not pass to the Einstein frame through a conformal transformation.

At first sight, the appearance of the non-geometrical, scalar field in the expression of entropy seems like a non-trivial obstacle to the entropy calculation in loop quantum gravity [12] because that approach is deeply rooted in quantum geometry. In a subsequent paper we will use the Hamiltonian framework developed here to pass to the quantum theory. One finds that the non-minimal coupling does introduce conceptual changes in the quantum geometry framework but one is again naturally led to a coherent description of quantum geometry.

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[3] If f vanishes in an open set, the symplectic structure at that point of the phase space acquires additional degenerate directions, i.e., the notion of 'gauge' is now enlarged. It would be interesting to analyze whether the evolution is unique modulo this extended gauge freedom even though, as pointed out in section II, the Cauchy problem is ill-posed in the standard sense.

The seamless matching between the isolated horizon boundary conditions, the bulk quantum geometry and the surface Chern-Simons theory which lies at the heart of the calculation of [12] continues but the statistical mechanical calculation now leads to the expression (1.2) of entropy.

Acknowledgments

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